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DYNAMICAL FINITE ELEMENT ANALYSIS FOR ELASTIC WAVES IN BEAMS AND PLATES

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Abstract—A special method of analysis, hereafter referred to as "Direct Analysis", is described and applied to the solution of the problem of traveling flexural waves in beams and plates, for which shear correction and rotatory inertia are considered. Finite beams and plates are considered so that the influence of reflected waves is included. The proper boundary conditions for these problems, several input stress pulses, as well as the effect of the length of the bounded medium (beam or plate) on the magnitude of the stress are considered. Implications to design of structures are discussed. The characteristic features of the Direct Analysis are then presented.

NOTATION

A	cross-sectional area
†A_i	cross-sectional area which contributes to dynamic inertia
A_s	cross-sectional area which contributes resistance to shearing
C_p	plate velocity $= [E/\rho(1-\nu^2)]^{\frac{1}{2}}$
C_1	dilatational wave velocity in a beam $= [EI_b/\rho I_i]^{\frac{1}{2}}$
C_2	shear wave velocity in a beam $= [A_s G/\rho A_i]^{\frac{1}{2}}$
C_2'	shear wave velocity in a plate $= [G/\rho]^{\frac{1}{2}}$
D	flexural rigidity of plate $= Eh^3/12(1-\nu^2)$
E	modulus of elasticity
G	modulus of rigidity $= E/2(1+\nu)$
h	plate thickness
I	cross-sectional moment of inertia
I_b	moment of inertia which contributes resistance to bending
I_i	moment of inertia which contributes resistance to dynamic inertia
j	superscript referring to quantities of the jth cell
j_m	number of elements (cells) into which beam or plate is divided
k	time for ramp to reach its maximum value
k_m	number of time intervals
k_s	shear correction factor for beam
k_2	shear correction factor for plate
l	length of beam
M	internal bending moment
M_r	radial bending moment per unit length
M_θ	tangential bending moment per unit length
Q_r	transverse shear force per unit length
q	intensity of distributed external load on beam
r	radial distance along plate

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† In this work subscripts are used as identifying symbols while superscripts indicate indices in the Direct Analysis.

r_0	inner radius of plate
r_l	outer radius of plate
S_B	slope of the deflection curve of a beam when shearing force is neglected
S_S	slope of deflection curve due to shear
t	time
V	vertical shear force on a cross-section of the beam
v	velocity of deflection in a beam or plate, y_t or w_t, respectively
w	transverse displacement of the midplane of plate
x	coordinate along length of beam
y	deflection of beam
β	angle of shear, measured at the neutral axis of a beam
ε_y	total slope of the deflection curve of the beam
ε_ψ	angular strain of an element of the beam
ε_ϕ	angular strain of an element of the plate
θ	tangential direction
ν	Poisson's ratio
ρ	density of the material of beam or plate
ϕ	rotation of the cross-section of the plate about the tangential axis
ψ	slope of the deflection curve of a beam when shearing force is neglected
ω	angular velocity of rotation of an element of the beam or plate, ψ_t or ϕ_t, respectively

1. INTRODUCTION

THIS paper treats the problem of traveling flexural stress waves in a finite beam and plate. By a traveling wave, we mean a disturbance (discontinuity in a function or its derivatives) which travels along the length of the medium without essential loss in shape. This disturbance travels at a constant speed in an elastic medium and its effects are felt by a stationary observer only after the arrival of the wave front. It has been shown [1] that in a problem dealing with flexural waves two distinct waves are generated regardless of the input pulse. This condition complicates the classical idea of a traveling wave since a coupling of the two waves occurs.

This work attempts to contribute a further understanding of these questions by solving problems of bounded media, i.e. media in which reflections are considered. As will be shown, the reflections considered in these problems give rise to far greater stresses than those encountered in semi-infinite media. For this reason, the latter analyses fall short in their applicability to problems of design.

Transverse and flexural impacts of beams and plates have long been recognized as problems of practical significance and importance. Several methods of solution have been investigated, such as Laplace Transforms and the method of characteristics. Mindlin [2] compared the elementary Euler–Bernoulli theory with the exact theory and with the Timoshenko beam theory. He pointed out that the shear correction, when applied in the Timoshenko theory, plays the largest single role in allowing the results of this theory to approach the results of the exact theory. Boley [3] and Dengler [4] used the Laplace Transform method to solve several boundary value problems using the Timoshenko beam theory. Miklowitz [5] and Lubkin [6] used this method in solving for the stresses in an infinite elastic plate due to a suddenly applied transverse load. Jahsman [7] applied the method of characteristics to the Uflyand–Mindlin equations of motion for a thin elastic plate. He derived the physical characteristics and characteristic equations. Leonard and Budiansky [8] successfully applied the method of characteristics to the Timoshenko beam equations. While their solutions are based on the assumption of equal wave velocities, they clearly show the nature of flexural waves. Chou [9, 10] successfully applied the method of characteristics to the solution of flexural stress wave problems in a semi-infinite circular plate due to several types of impulse loadings.

Recently, Kelly [11] analyzed wave propagation effects on a Timoshenko beam for the purpose of estimating initial deceleration for a mass impact.

The method of analysis employed herein is the Direct Analysis which has been successfully employed by Davids [12], Mehta [13] and Davids [14, 15] in the solution of problems of cylindrical and spherical elastic and elastoplastic waves as well as problems such as those that are treated in this work. This method is extended herein to include a technique which treats the coupling of the two waves arising as a result of a transverse impact on a beam or plate.

A numerical technique, similar to Direct Analysis, known as Finite Element Analysis has been extensively studied by J. H. Argyris (Imperial College) and O. C. Zienkiewicz (Swansea). In addition, J. R. H. Otter [16] has recently developed a numerical technique known as Dynamic Relaxation which is related to Finite Element Analysis.

The beam and plate are treated in this paper in a unified manner since the basic nature of their governing equations is similar.

2. STATEMENT OF THE PROBLEM

2.1 *Physical laws*

The assumptions which are necessary to derive the physical laws relating to the Timoshenko beam may be found in [17]. Analogous assumptions are used for the circular plate.

2.1.1 *Timoshenko beam.* We begin our analysis by dividing the length of the beam into j_m elements of equal length dx. The equations of motion for the Timoshenko beam may be obtained by applying the impulse–momentum laws to a typical jth element of the beam (Fig. 1), viz

Rotation of the jth element

$$d\omega = \frac{\{\frac{1}{2}(V^j + V^{j+1})\,dx + M^j - M^{j+1}\}\,dt}{\rho I_i\,dx} \tag{1}$$

Translation of the jth element

$$dv = \frac{(V^{j+1} - V^j + q\,dx)\,dt}{\rho A_i\,dx} \tag{2}$$

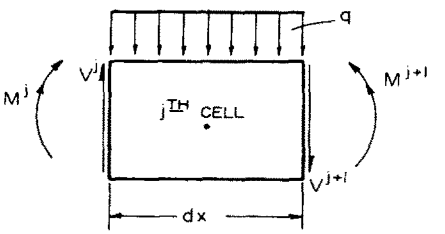

FIG. 1. Free-body diagram of beam element (typical jth element).

The constitutive relations for the Timoshenko beam may be obtained upon consideration of the deformations of a typical beam element (Fig. 2).

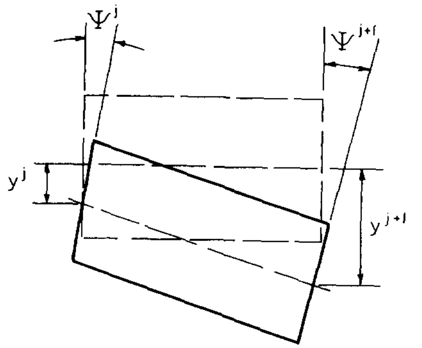

FIG. 2. Deformations of an element of beam.

$$V = A_s G\{\varepsilon_y - \psi\} \tag{3}$$

$$M = -EI_b\{\varepsilon_\psi\} \tag{4}$$

where

$$\varepsilon_y = (y^{j+1} - y^j)/\mathrm{d}x \tag{3a}$$

$$\varepsilon_\psi = (\psi^{j+1} - \psi^j)/\mathrm{d}x \tag{4a}$$

The dilatation wave velocity C_1 with which discontinuities in ω and M, as well as discontinuities in their higher derivatives, propagate may be shown to be [8]

$$C_1 = \left(\frac{EI_b}{\rho I_i}\right)^{\frac{1}{2}} \tag{5}$$

Similarly, the shear wave velocity C_2 with which discontinuities in v and V, as well as discontinuities in their higher derivatives, propagate may be shown to be

$$C_2 = \left(\frac{A_s G}{\rho A_i}\right)^{\frac{1}{2}} \tag{6}$$

Any input to the beam which causes it to undergo a flexural mode of motion immediately produces waves of both velocities C_1 and C_2. This means that even a step moment input to the beam, for example, produces two traveling waves which propagate from the point of impact. The magnitudes of these velocities are seen to be constant.

It may be shown [8] that a discontinuity in a moment (step moment pulse) produces a discontinuity in M and ω. These discontinuities travel along the length of the beam at C_1. In addition, this input produces discontinuities in higher derivatives of v and V which travel at C_2. There are no discontinuities in the values of v and V themselves when a step moment is applied at the end of a beam. Thus, a discontinuity in a quantity that is generated across one wave front immediately generates a discontinuity in a higher derivative of the quantities associated with the opposite wave. In this way, the coupling of all physical quantities becomes apparent. The fact that coupling exists is also apparent from the equations (1)–(4).

2.1.2 *Circular plate with shear correction and rotatory inertia.* The assumptions necessary to derive the physical laws of the circular elastic plate in which shear correction and rotary inertia are considered are analogous to those used in the beam derivations above.

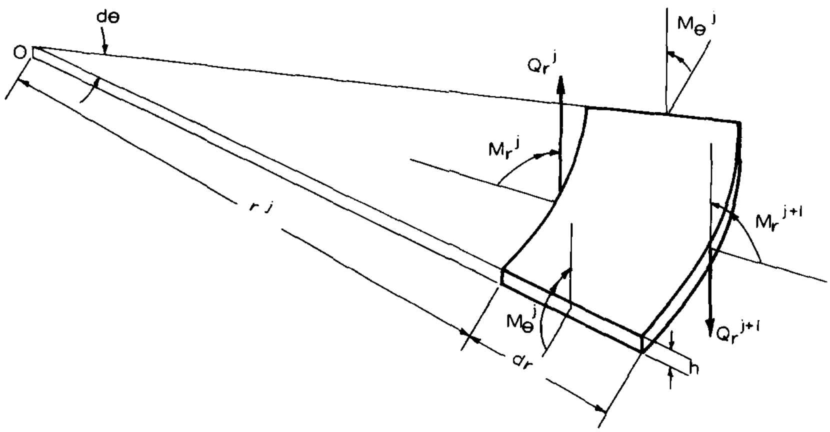

FIG. 3. Free-body diagram of plate element (typical jth element).

In a similar manner as above, the plate is divided into j cells (Fig. 3). The equations of motion become

Rotation of a jth cell

$$d\omega = \frac{\{M_r^{j+1}(r^j+\mathrm{d}r)-r^jM_r^j-M_\theta^j\,\mathrm{d}r-\frac{1}{2}(r^jQ_r^j\,\mathrm{d}r+[r^j+\mathrm{d}r]Q_r^{j+1}\,\mathrm{d}r)\}\,\mathrm{d}t}{(\rho h^3/12)(r^j\,\mathrm{d}r+\mathrm{d}r^2/2)} \tag{7}$$

Translation of the jth cell

$$\mathrm{d}v = \frac{[(r^j+\mathrm{d}r)Q_r^{j+1}-r^jQ_r^j]\,\mathrm{d}t}{\rho h(r^j\,\mathrm{d}r+\mathrm{d}r^2/2)} \tag{8}$$

The constitutive relations are

$$M_r = D\left\{\varepsilon_\phi+\frac{v}{r}\phi\right\} \tag{9}$$

$$M_\theta = D\left\{\frac{1}{r}\phi+v\varepsilon_\phi\right\} \tag{10}$$

$$Q_r = k_2^2Gh\{\phi+\varepsilon_w\} \tag{11}$$

where:

$$\varepsilon_\phi = (\phi^{j+1}-\phi^j)/\mathrm{d}r \tag{10a}$$

$$\varepsilon_w = (w^{j+1}-w^j)/\mathrm{d}r \tag{11a}$$

The dilatation wave velocity C_p (also referred to as plate velocity) and the shear wave velocity $k_2 C_2'$ are, respectively

$$C_p = \left[\frac{E}{\rho(1-\nu^2)}\right]^{\frac{1}{2}} \tag{12}$$

$$k_2 C_2' = k_2\left[\frac{G}{\rho}\right]^{\frac{1}{2}} \tag{13}$$

where k_2^2 may be expressed as [5]

$$k_2^2 = 0{\cdot}76 + 0{\cdot}3\nu. \tag{14}$$

2.2 *Boundary conditions*

In the problems which are solved in this work, the formulation of the boundary conditions must be accomplished with the aid of the special assumptions of the Timoshenko beam theory and the analogous assumptions of the shear corrected plate. In the Timoshenko beam the total slope of an element is given by [17]

$$\begin{aligned} S_T &= S_B + S_S \\ \varepsilon_y &= \psi + \beta \end{aligned} \tag{15}$$

For the problem solved herein, namely the cantilever beam, the appropriate boundary conditions are

$$y|_{x=l} = 0 \tag{16}$$

$$\psi|_{x=l} = S_B|_{x=l} = 0 \tag{17}$$

We note that equation (17) specifies that the so-called "bending slope" ψ is zero at the fixed end. The total slope, however, is not zero as it would be in the simpler Euler–Bernoulli theory. This condition occurs since the cross-section may undergo a "shear rotation" at the fixed end of the beam. In our problem, it was convenient to express the boundary conditions in terms of the linear and angular velocities, respectively

$$v|_{x=l} = y_t|_{x=l} = 0 \tag{18}$$

$$\omega|_{x=l} = \psi_t|_{x=l} = 0 \tag{19}$$

Analogous boundary conditions may be stated for the plate with its outer edge clamped, viz.

$$v|_{r=r_l} = w_t|_{r=r_l} = 0 \tag{20}$$

$$\omega|_{r=r_l} = \phi_t|_{r=r_l} = 0 \tag{21}$$

Relations (1)–(6) along with equations (18) and (19) are a complete statement of the problem of the traveling flexural stress waves in a cantilever beam. Relations (7)–(14) in addition to equations (20) and (21) formulate the analogous problem for the plate. These relations may be used directly in a computer code.

3. DIRECT ANALYSIS OF A CANTILEVER BEAM

The Direct Analysis formulation will now be presented for the cantilever beam. For the sake of brevity, a similar analysis for the circular plate will be omitted here because it is entirely analogous.

The Direct Analysis is a numerical method of solution which bypasses the derivation and use of differential equations as unnecessary steps in solving the problem under consideration. Instead, the actual statements of the governing physical laws are essentially all that is necessary to effect a solution. These laws are applied directly to a finite system of elements. It must be pointed out here that Direct Analysis is *not* a form of the numerical procedure known as "finite-differences". In the latter, a governing differential equation is first derived by mathematical techniques and only then is the numerical "finite-difference" method applied. The physical laws (impulse–momentum, constitutive relations) remain in their original form in this analysis.

The analysis is begun by dividing the medium under consideration, e.g. beam or plate, into a finite number of cells. The time increment is determined from $dt = dx/C$, [18] where C is the dilatational wave velocity for the problem under consideration. The cell size is thus arbitrary and is varied until any reduction in this quantity will not yield any significant change in the resulting solution of the problem.

Sequential summary of the direct analysis for a cantilever beam

(i) Specify given data: $M(t)$, $V(t)$, ρ, E, l, v, dx, k_s, k_m, I_i, I_b, A_i, $q(t)$

(ii) Define:

$$G = E/2(1+v) \tag{22a}$$

$$A_s = k_s A_i \tag{22b}$$

$$C_1 = [EI_b/\rho I_i]^{\frac{1}{2}} \tag{22c}$$

$$C_2 = [A_s G/\rho A_i]^{\frac{1}{2}} \tag{22d}$$

$$dt = dx/C_1 \tag{22e}$$

$$j_m = l/dx \tag{22f}$$

$$x^{j+1} = x^j + dx \tag{22g}$$

(iii) *Pulse inputs*

$$M(0, t) = M(t) \tag{23a}$$

$$V(0, t) = V(t) \tag{23b}$$

$$q = q(t) \tag{23c}$$

(iv) *Propagation procedure (dilatation wave)*: $j = 1, 2, \ldots, j_m$

$$d\varepsilon_\psi = (\omega^{j+1} - \omega^j)\, dt/dx \tag{24a}$$

$$dM = -EI_b\, d\varepsilon_\psi \tag{24b}$$

$$(M^{j+1})' = M^{j+1} + dM^* \tag{24c}$$

* Relation (24c) and succeeding ones of the same form are cumulation operations which add the incremental quantity to the value of the variable which previously exists.

$$(\psi^j)' = \psi^j + \omega^j \, dt \tag{24d}$$

$$d\omega = \frac{[\frac{1}{2}(V^j + V^{j+1})\, dx + M^j - M^{j+1}]\, dt}{\rho I_i \, dx} \tag{24e}$$

$$(\omega^j)' = \omega^j + d\omega \tag{24f}$$

$$\omega^{j_m+1} = 0 \tag{24g}$$

(v) *Propagation procedure* (*shear wave*): $j = 1, 2, \ldots, j_m$

$$d\varepsilon_y = (v^{j+1} - v^j)\, dt/dx \tag{25a}$$

$$dV = A_s G\{d\varepsilon_y - \omega^{j+1}\, dt\} \tag{25b}$$

$$(V^{j+1})' = V^{j+1} + dV \tag{25c}$$

$$(y^j)' = y^j + v^j \, dt \tag{25d}$$

$$dv = \frac{(V^{j+1} - V^j + q\, dx)\, dt}{\rho A_i \, dx} \tag{25e}$$

$$(v^j)' = v^j + dv \tag{25f}$$

$$v^{j_m+1} = 0 \tag{25g}$$

(vi) *Increment time*

$$(t)' = t + dt \tag{26}$$

Repeat steps (iii)–(vi) for a specified number of times k_m.

An entirely analogous analysis may be performed for the plate. We note that there is no specific operation or expression which specifically deals with a forward or reflected traveling wave i.e., these are not separately identified. If a step moment, for example, is applied at cell 1, the solution obtained from the Direct Analysis shows a sharp wave front traveling across the medium with the proper velocity. The conditions (24g) and (25g) at the terminal cell are sufficient to generate a reflected wave. This reflected wave then propagates in the reverse direction and is superimposed on the incident wave.

4. DISCUSSION OF RESULTS

The Direct Analysis is applied to several problems and the accuracy of the solution is checked in the case of the ramp moment input to a circular plate with its outer edge clamped. The problems solved herein are

(A) Ramp Moment on Beam—l = 1·5 in.
(B) Step Moment on Beam—l = 1·5 in.
(C) Step Moment on Beam—l = 3·0 in.
(D) Ramp Shear on Beam—l = 1·5 in.
(E) Static Solution for Moment Input to Beam—l = 2·0 in.
(F) Ramp Moment on Plate—r_0 = 0·25 in., r_l = 0·85 in.

Unless otherwise specified, the constants used in these solutions are

Beam: $E = 30 \times 10^6$ lb/in^2, $\nu = 0{\cdot}3$, $\gamma = g\rho = 0{\cdot}300$ lb/in^3
$A = 1{\cdot}00$ in^2, $k_s = 0{\cdot}833$, $I_b/I_i = 1{\cdot}0$
Plate: $E = 28 \times 10^6$ lb/in^2, $\nu = 0{\cdot}3$, $\gamma = g\rho = 0{\cdot}286$ lb/in^3
$h = 0{\cdot}125$ in.

The solutions presented herein are shown to demonstrate the generality of the present technique and the behavior of reflected traveling wave solutions. Parameters such as length and area are inputs to the analysis and thus may be of any value which is consistent with the problem under consideration. The particular numerical dimensions used were chosen to illustrate the technique of analysis only.

(A) *Ramp moment input to a cantilever beam—$l = 1{\cdot}5$ in.*

Figures 4 and 5, respectively, show the internal bending moment and shear that are induced at position $x = 0{\cdot}5$ in. by a ramp moment input at $x = 0$. The rise time of the ramp is given by $k = 5{\cdot}174$ μsec as shown. The discontinuities in the derivatives of the moment (Fig. 4) at $t = 8$ μsec and at $t = 13$ μsec are the manifestations of the discontinuity in the derivative of the input pulse. If the beam were semi-infinite in length, the bending moment, after reaching its initial maximum, would decay and approach the static value of 1. However, the wave front which is reflected from the fixed end of the cantilever beam arrives at $x = 0{\cdot}5$ in. at $t = 13$ μsec. After this time, the value of the moment is seen to oscillate with a large amplitude. At $t = 42$ μsec, the value of the moment is seen to be considerably higher than that at $t = 15$ μsec. This phenomenon clearly shows that the effects of the reflections are to increase the level of the stresses. For this reason, reflections must be considered in order to obtain accurate design data for the stresses. This observation is even more vividly demonstrated in Fig. 5. The static solution for this problem demands that the shear force V approach zero. If the medium were semi-infinite in length, this would indeed be the case.

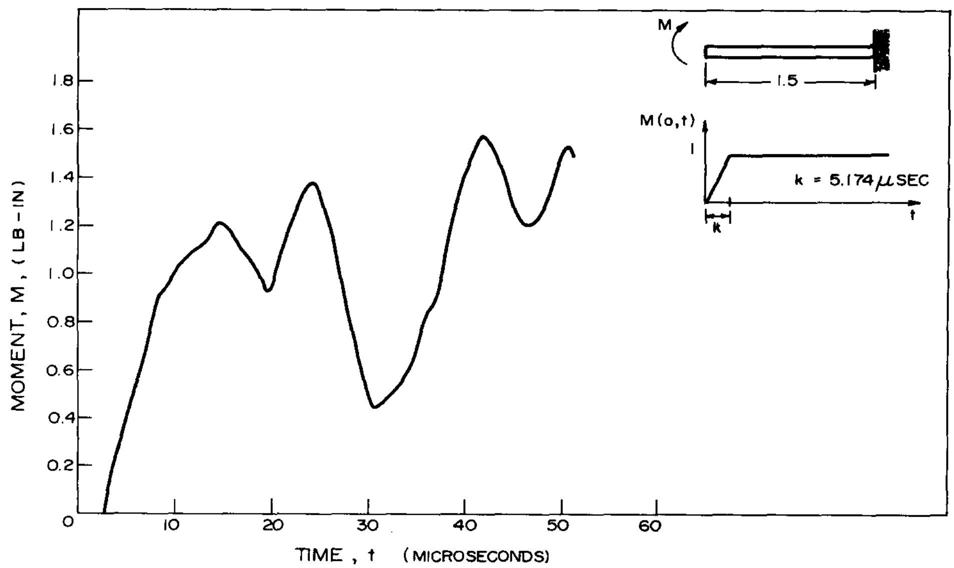

FIG. 4. Bending moment, M, vs. time, t; $x = 0{\cdot}5$ in.

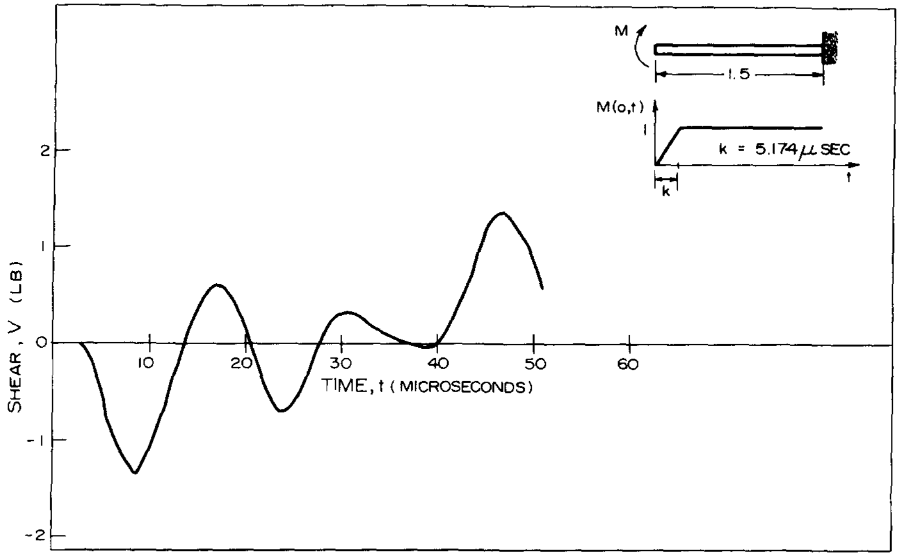

FIG. 5. Shear, V, vs. time, t; $x = 0{\cdot}5$ in.

However, the reflected waves cause the shear to take on values other than zero as time increases.

(B) *Step moment input to a cantilever beam—$l = 1{\cdot}5$ in.*

The moment and shear shown in Figs. 6 and 7, respectively, are induced at $x = 0{\cdot}5$ in. by a step moment input at $x = 0$. The discontinuities in the moment curve shown at $t = 2{\cdot}59$, 13, 18·18, ... μsec, depict, respectively, the effects of the arrival of the outward-going wave front and its successive reflections. This case is more severe than the ramp moment. One may clearly see that the effect of reflections on the magnitude of the moment is to produce spikes. For example, as one may see from Fig. 6, the moment dynamic intensification factor $M_{\text{dyn}}/M_{\text{stat}} = 2{\cdot}24$ for the cantilever beam under consideration. If the beam were semi-infinite in length the moment dynamic intensification factor would be 1·15. In addition, one may note that several reflections were required before this peak moment value was attained. Thus, in problems relating to traveling stress waves, several reflected wave fronts must be observed before an estimate of the maximum stress levels in a medium can be obtained. These delays in the maximum stress levels are due to the dispersion in the medium. It is again worthwhile to note the importance of considering a bounded medium as opposed to a semi-infinite one.

(C) *Step moment input to a cantilever beam—$l = 3{\cdot}0$ in.*

In order to study how the length of the beam affects the dynamic behavior of the moments and shears, a beam length of 3 in. was investigated. This is twice the length used in the previous case. In Figs. 8 and 9, the moment and shear, respectively are shown for this configuration. The position where these variables were monitored was $x = 0{\cdot}5$ in. as in the previous cases. One notes that the moment and shear are approaching their static values of 1 and 0, respectively, until the time of arrival of the reflected wave front. Upon

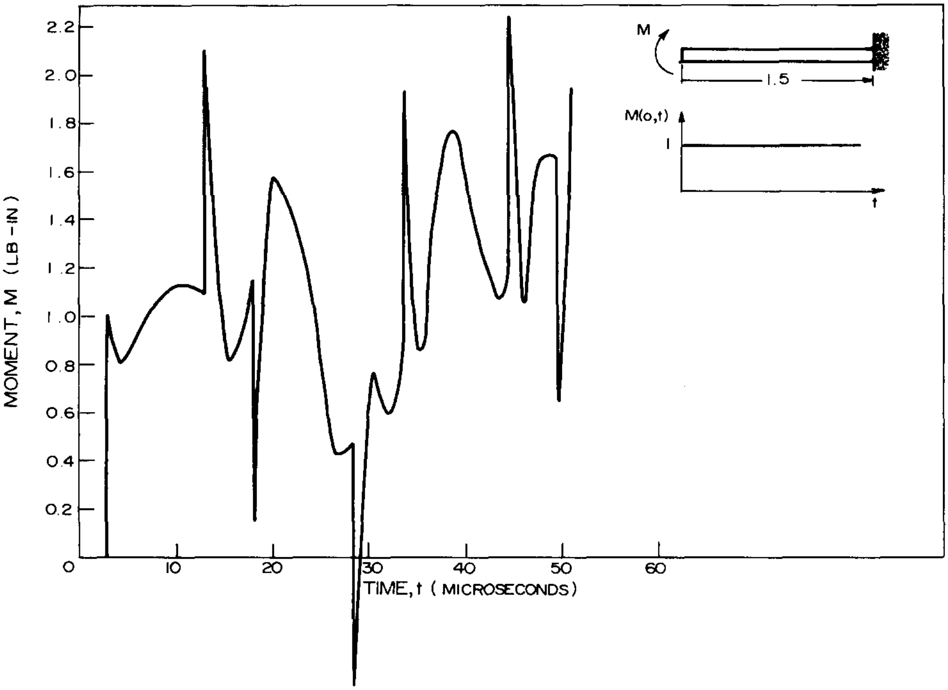

FIG. 6. Bending moment, M, vs. time, t; $x = 0{\cdot}5$ in.

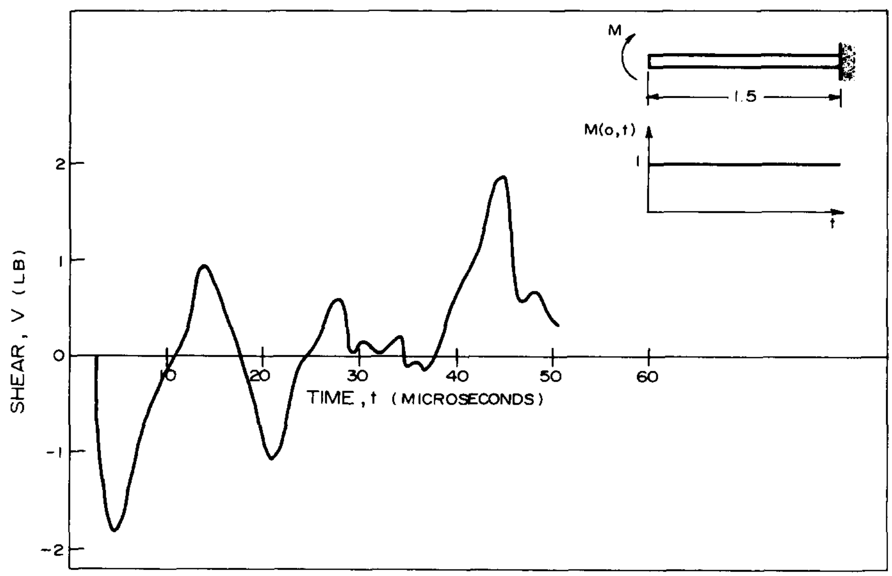

FIG. 7. Shear, V, vs. time, t; $x = 0{\cdot}5$ in.

comparing the moment and shear of this problem with those of the preceding problem, one notes that the increase of beam length has the effect of reducing the severity of the

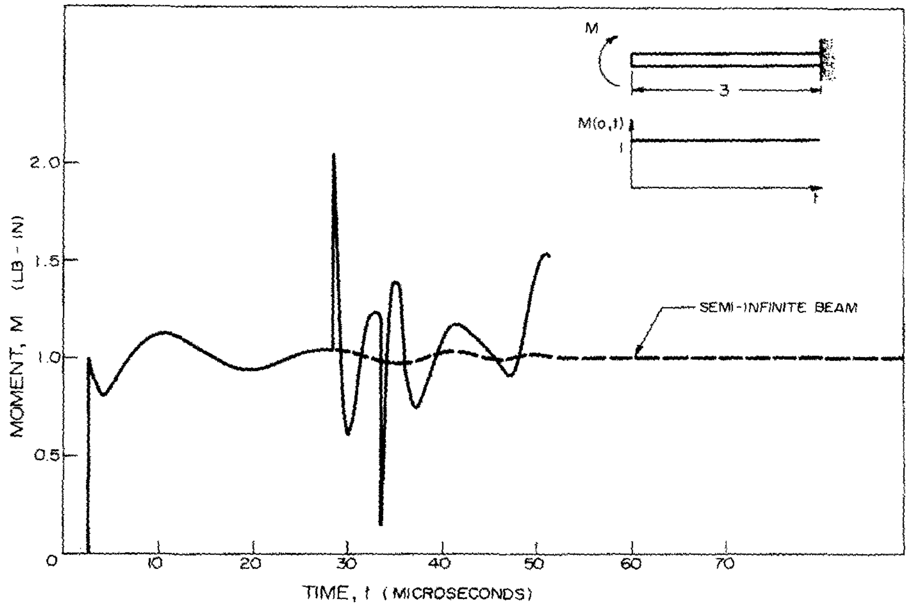

FIG. 8. Bending moment, M, vs. time, t; $x = 0{\cdot}5$ in.

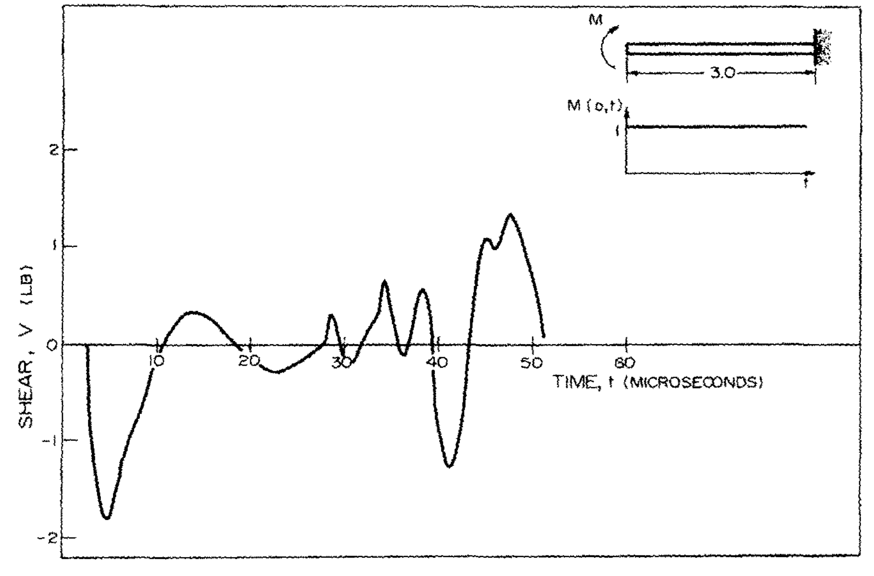

FIG. 9. Shear, V, vs. time, t; $x = 0{\cdot}5$ in.

stress levels. This is as expected since, if l approaches infinity, the maximum bending moment would occur at 10 μsec and, thereafter, approach its static value. Thus, one notes that dynamic intensification factors of the beam are functions of the length. The solution which would be obtained for this problem, were the beam semi-infinite in length, is shown by the dotted line in Fig. 8. One immediately notices that the stress levels predicted by the latter configuration would be in error.

(D) *Ramp shear on beam*—$l = 1{\cdot}5$ *in.*

Figures 10 and 11, respectively, show the bending moment and shear force at $x = 0{\cdot}5$ in. for a ramp shear input to a cantilever beam of length $l = 1{\cdot}5$ in. One may particularly note

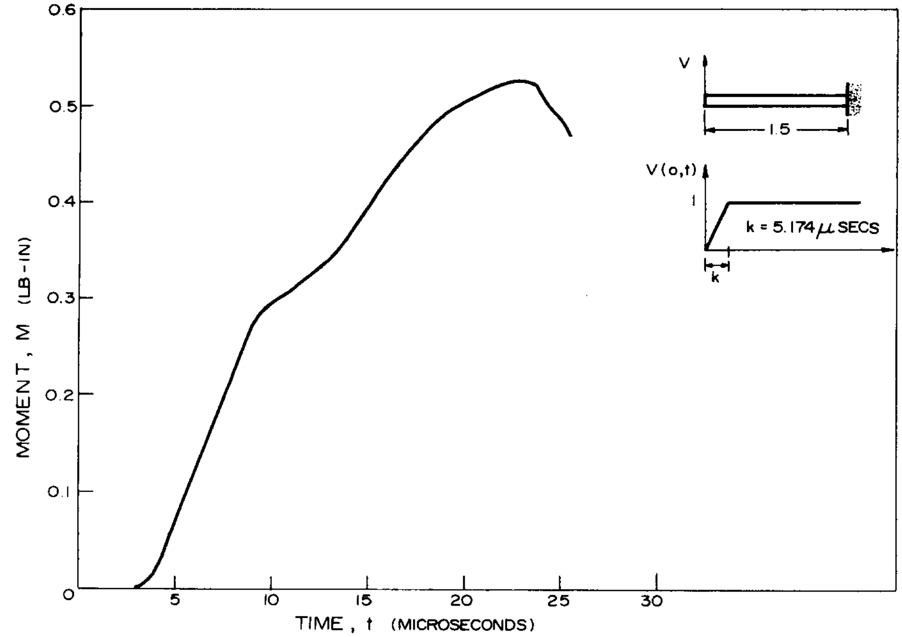

FIG. 10. Bending moment, M, vs. time, t; $x = 0{\cdot}5$ in.

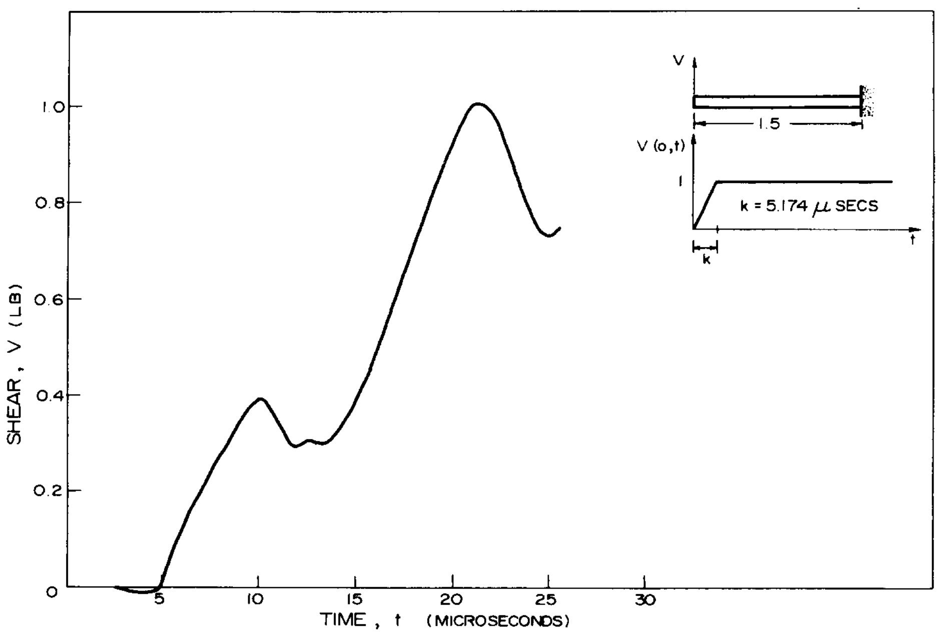

FIG. 11. Shear, V, vs. time, t; $x = 0{\cdot}5$ in.

the dynamic behavior of the shear force. When the beam is impacted by the ramp shear, both a dilatation and a shear wave are generated. Upon arrival of the dilatation wave at $x = 0{\cdot}5$ in., the shear force commences to become negative even though an impulse of $+1{\cdot}0$ is input at $x = 0$. When the shear wave arrives, the shear force becomes positive as required by the input pulse. Due to the necessity of computational accuracy in this problem, a smaller mesh size was necessary than in the previous problems. For this reason, the solution was only carried out to $t = 25$ μsec which was considered most efficient. The solution may, however, be carried out to any arbitrary time.

(E) *Static solution for moment input to a cantilever beam*—$l = 2{\cdot}0$ *in.*

Figures 12 and 13, respectively, show the moment and shear that are induced at positions $x = 0{\cdot}5$ in. and $x = 1{\cdot}5$ in. when a ramp moment input is applied to a cantilever beam of length 2 in. The rise time of the ramp is 10·34 μsec. In this problem, however, the angular and linear velocities, respectively, are exponentially damped so that a static solution may be obtained, viz

$$\omega = \omega_0 e^{-t/\tau_1}$$

$$v = v_0 e^{-t/\tau_2}$$

where τ_1 and τ_2 are arbitrary damping constants. In this problem $\tau_1 = \tau_2 = 1{\cdot}2 \times 10^{-5}$ sec was used. One may note that the static values of 1 and 0 for the moment and shear, respectively, are obtained as time increases. Points A and B indicate the time at which the moment at positions $x = 0{\cdot}5$ in. and $x = 1{\cdot}5$ in., respectively, reach their static values to two decimal places. Since these times are dependent upon the values of τ_1 and τ_2 they are not significant. The important point to consider is the fact that this solution was obtalned with the identical formulation and analysis as all the other solutions in this paper. It is generally not possible to attain both the static and the dynamic solutions of a problem with many other techniques of solution. A further point to consider is the fact that the static solution obtained in this way provides the deflection and rotation with the shear contribution to these values present. In elementary textbooks, the shear contribution to the deflection and rotation of a beam is usually disregarded since its inclusion provides great difficulties in the analysis. If experimental results were available for the values of the damping constants τ_1 and τ_2, they could be inserted into the analysis. This would then yield a dynamic behavior which would closely approximate the actual physical behavior of the medium. Conversely, if the dynamic behavior of the beam were known, our analysis could be used to test for compatible values of these constants.

(F) *Ramp moment on plate*—$r_0 = 0{\cdot}25$ *in.*, $r_l = 0{\cdot}85$ *in.*

Figures 14 and 15 show the radial moment and transverse shear, respectively that are induced at $r = 0{\cdot}375$ in. by a ramp moment of $k = 1{\cdot}227$ μsec that is applied at the inner edge of a circular plate whose outer edge is clamped. One again notes that the effect of the outer boundary is to increase the moments and the shear because of its reflection capability. This solution is compared (large points) with a similar one for a ramp moment input to a circular plate which is semi-infinite in length. The solution to the latter problem had previously been accomplished by the method of characteristics [9]. In the characteristics solution a mesh size of $dr = 0{\cdot}003125$ in. was used. A larger mesh size for this problem

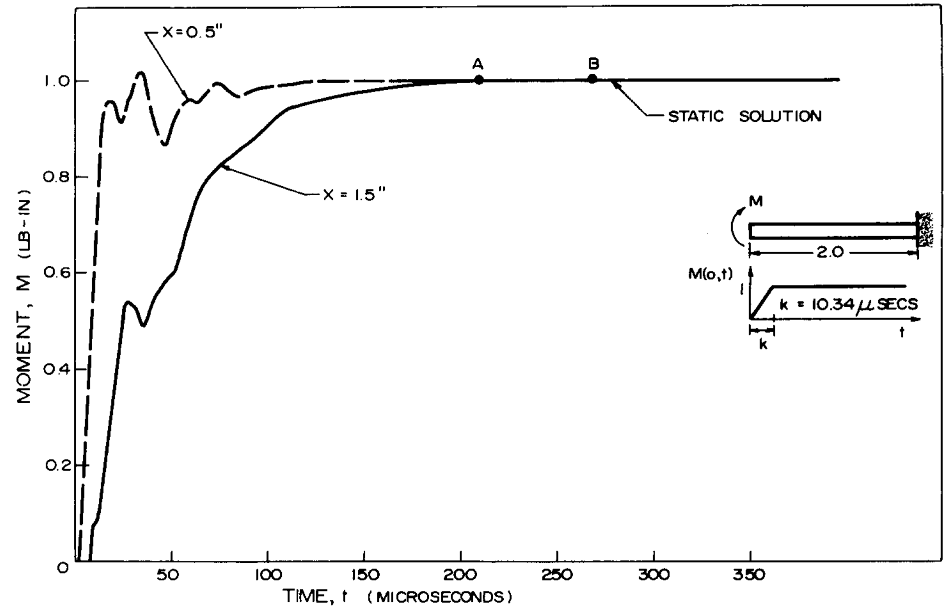

FIG. 12. Bending moment, M, vs. time, t; $x = 0{\cdot}5$ in. and $x = 1{\cdot}5$ in. ($\tau_1 = \tau_2 = 1{\cdot}2\times10^{-5}$ sec).

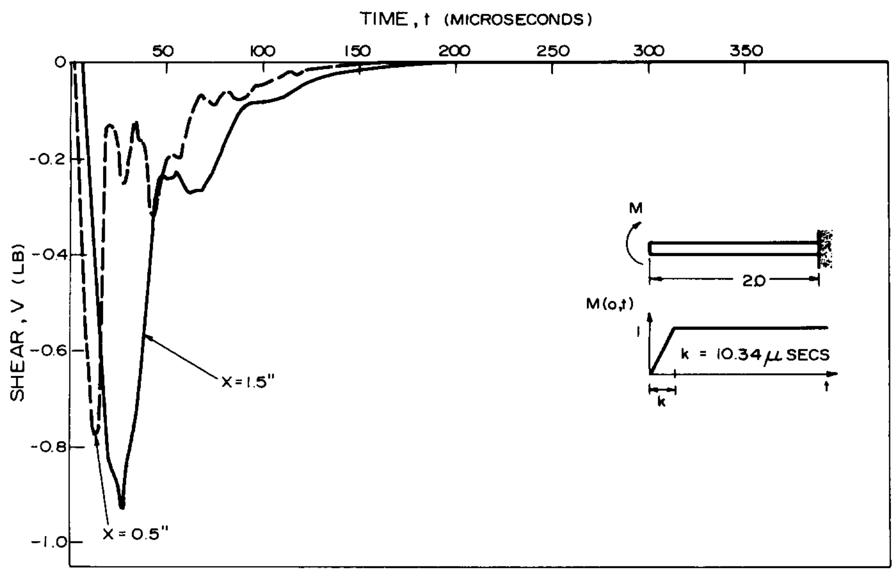

FIG. 13. Shear, V, vs. time, t; $x = 0{\cdot}5$ in. and $x = 1{\cdot}5$ in. ($\tau_1 = \tau_2 = 1{\cdot}2\times10^{-5}$ sec).

with the method of characteristics produces an inaccurate solution. Only upon reducing the mesh size to 0·003125 in. does that solution converge to one with proper accuracy. However, the mesh size used in the Direct Analysis to obtain this solution was $dr = 0{\cdot}0125$ in. which is four times as coarse. The mesh size of the Direct Analysis solution was also varied

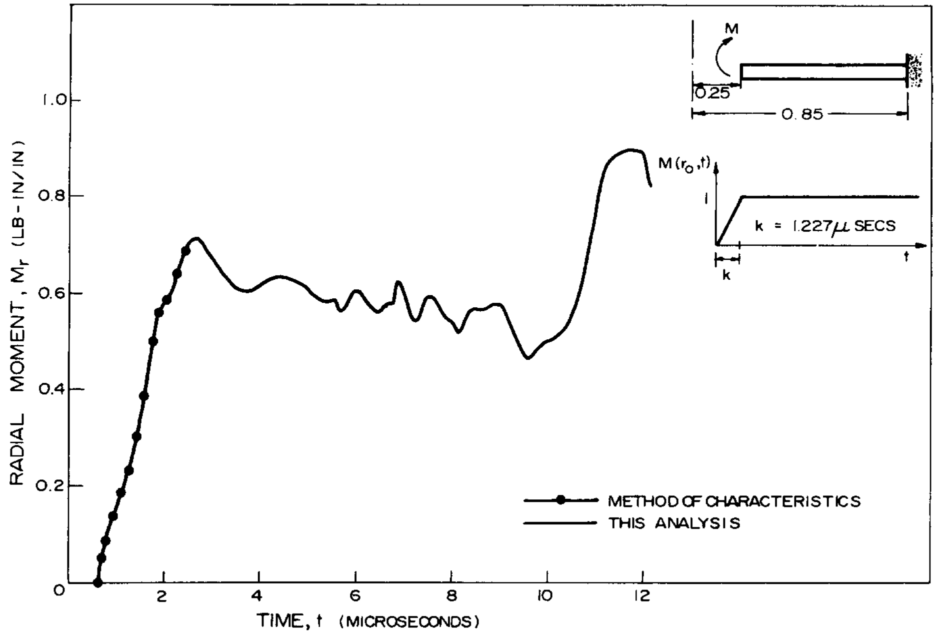

FIG. 14. Radial moment, M_r, vs. time, t; $r = 0{\cdot}375$ in.

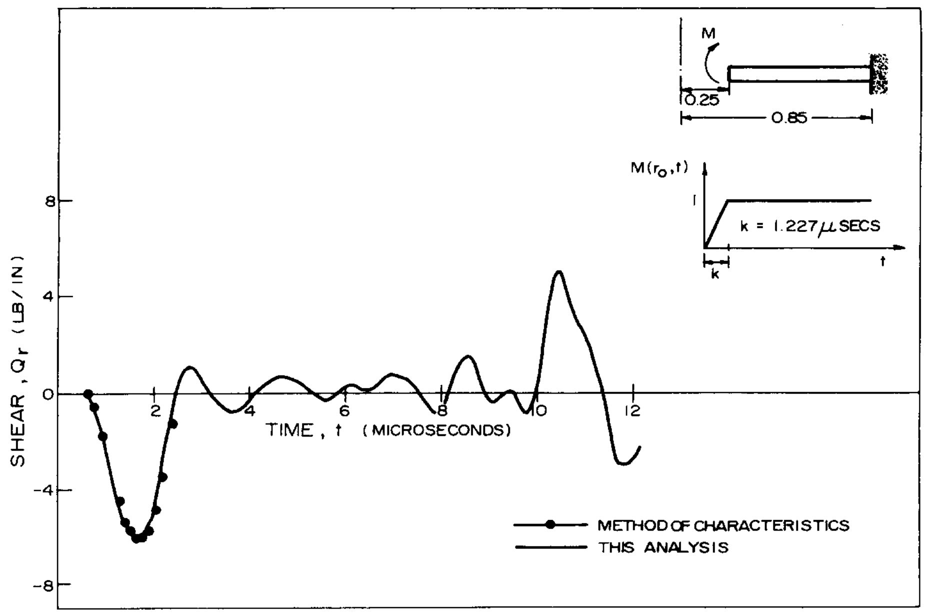

FIG. 15. Shear, Q_r, vs. time, t; $r = 0{\cdot}375$ in.

until satisfactory convergence was obtained with $dr = 0{\cdot}0125$ in. Thus, for this problem, the Direct Analysis may be seen to be more efficient.

The reason that the Finite Element Analysis tends to be more accurate than the method of characteristics for this class of problems is that in the latter technique finite-difference "averaging" of the dynamic quantities is necessary to numerically integrate the characteristic equations. No such averaging is necessary with the present technique.

Error accumulation, which is a difficulty in any numerical method, was negligible in the problems solved in this work. This remains the case until an excessive number of time increments are employed. For example, in a problem previously solved by the Direct Analysis, errors become evident only after 600 μsec. The time interval in that problem was $\frac{1}{4}$ μsec. Thus, only after 2400 time intervals did errors begin to become evident. Since, in problems of flexural impact, the critical dynamic history generally occurs in less than 100 μsec, error accumulation does not appear to be a significant problem within the scope of this analysis. If longer time intervals are required for a problem, it is generally better to use a technique other than a wave approach.

In this work, comparison has been made with other partial solutions developed by numerical techniques. A comparison of our solutions with those which have been developed from exact analyses is given in [15].

Additional solutions to problems of the flexural traveling waves in beams and plates which are finite in length may be found in [14] and [15]. A more detailed presentation of the Direct Analysis of the plate is also given in [14]. The Direct Analysis presented herein has been developed into a computer code in order to obtain the solutions presented above. The solutions were obtained on the IBM 7074 Digital Computer at The Pennsylvania State University Computation Center.

5. CONCLUSIONS

The Direct Analysis as applied to a beam or plate in which shear deformation and rotatory inertia have been considered exhibits the following features:

(a) A bounded medium, i.e. one in which reflections are considered, offers no additional complexity than a non-bounded (semi-infinite) one. Reflections automatically occur upon statement of the proper boundary conditions.

(b) The solution of a problem in which boundaries and damping are present closely approximates the actual physical conditions and gives more realistic results than the solution of a semi-infinite medium. In this manner realistic design data and criteria may be obtained.

(c) The static solution to any of the problems which are illustrated has been obtained by damping both the angular and linear velocities until they vanish as explained above. This damping formulation is inserted in addition to the impulse–momentum laws. In this manner both the dynamic ($\tau \to \infty$) and the static solutions may be obtained. This may further be viewed as an alternate means of obtaining a static solution. This procedure of obtaining a static solution has the advantage over many of the procedures employing mathematical solutions since in the latter, dynamic and static analyses must be undertaken separately.

(d) The size of the cell required for an accurate solution is, in general, far coarser than that required with other numerical techniques. This observation is based on past and present work involving the Direct Analysis. (See for example, [12–15]). As has previously been shown for the ramp moment input to the plate, the present method achieved the same result for this problem with coarser mesh than other methods.

REFERENCES

[1] H. N. ABRAMSON, H. J. PLASS and E. A. RIPPERGER, Stress wave propagation in rods and beams. *Adv. appl. Mech.* **5**, 111 (1958).

[2] R. D. MINDLIN, Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. *J. appl. Mech.* **18**, 31–38 (1951).

[3] B. A. BOLEY and C. C. CHAO, Some solutions of the Timoshenko beam equations. *J. appl. Mech.* **22**, 579–586 (1955).

[4] M. A. DENGLER and M. GOLAND, Transverse impact of long beams including rotatory inertia and shear effects. *Proc. 1st U.S. natn. Congr. appl. Mech.*, 1952, pp. 179–186.

[5] J. MIKLOWITZ, Flexural stress waves in an infinite elastic plate due to a suddenly applied concentrated transverse load. *J. appl. Mech.*, 681–689 (Dec. 1960).

[6] J. L. LUBKIN, Propagation of elastic impact stresses. *ONR progr. Rep.* No. 5, Contract No. Nonr-704(00) (April 1954).

[7] W. E. JAHSMAN, Propagation of abrupt circular wave fronts in elastic sheets and plates. *Proc. 3rd natn. U.S. Congr. appl. Mech.*, 1958, pp. 115–202.

[8] R. W. LEONARD and B. BUDIANSKY, On traveling waves in beams. *NACA tech. Rep.* 1173 (1954).

[9] P. C. CHOU and H. A. KOENIG, Flexural waves in elastic circular plates by method of characteristics. *D.I.T. Rep.* No. 160-6 (Aug. 1965).

[10] P. C. CHOU, Flexural wave propagation in a circular plate due to impulsive loads. *Proc. 6th Int. Symp. Space Technology and Science*, Dec. 1965.

[11] J. M. KELLEY, The impact of a mass on a beam. *Int. J. Solids Struct.* **3**, 191–196 (1967).

[12] N. DAVIDS, P. K. MEHTA and O. T. JOHNSON, Spherical elasto-plastic waves in materials. *Behavior of Materials under Dynamic Loading.* ASME (1965).

[13] P. K. MEHTA and N. DAVIDS, A direct numerical analysis method for cylindrical and spherical elastic waves. *AIAA Jnl* **4**, 112–117 (1966).

[14] N. DAVIDS and H. KOENIG, Direct analysis of the flexural traveling waves in beams and plates. *Engng Mech. Bull.* No. 1 (Sept. 1966).

[15] N. DAVIDS and H. KOENIG, Double stress-wave discontinuities in finite, shear corrected beams and plates. *10th Midwestern Mechnics Conf.*, Aug. 1967. To be published in *Developments in Mechanics*, Vol. IV.

[16] J. R. H. OTTER, Dynamic relaxation. *Proc. Instn civ. Engrs* **35**, 633–656 (1966).

[17] S. TIMOSHENKO, *Vibration Problems in Engineering*, 3rd edition, pp. 329–331. Van Nostrand (1955).

[18] R. COURANT, K. FRIEDRICHS and H. LEWY, Über die Partiellen Differenzengleichungen der Mathematischen Physik. *Math. Annln* **100**, 32–74 (1928).

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Абстракт—Исследуется специальный метод анализа, названного в данном случае "Простым методом", применимым к решению задачи странствующих изгибных волн в балках и пластинках, для которых учитывается поправка от сдвига и моменты вращения. Исследуются конечные балки и пластинки, и поэтому учитивается влияние отраженных волн. Рассматриваются соответствующие граничные условия этих задач, отдельные входные импульсы напряжений, а также эффект длины ограниченого тела /балка или пластинка/ на величину напряжений. Обсуждаются применения метода к расчету конструкций. Приводятся характеристические детали "Простого метода".